

# EXISTENCE AND UNIQUENESS OF OPTIMAL TRANSPORT MAPS

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ABSTRACT. Let  $(X, d, m)$  be a non-branching Polish metric measure space. We show existence and uniqueness of optimal transport maps for cost written as non-decreasing and strictly convex functions of the distance, provided the space satisfies the measure contraction property.

## 1. INTRODUCTION

In [10], Gaspard Monge studied the by now famous minimization problem

$$(1.1) \quad \min_{T_{\#}\mu_0=\mu_1} \int d(x, T(x))\mu_0(dx),$$

on Euclidean space. This problem turned out to be very difficult because the functional and the corresponding set over which we minimize may both have a non-linear structure. 70 years ago, Kantorovich [8] came up with a relaxation of the minimization problem (1.1). He allowed arbitrary couplings of the two measures  $\mu_0$  and  $\mu_1$  and also more general cost functions  $c : X \times X \rightarrow \mathbb{R}$  are allowed:

$$(1.2) \quad \min_{q \in \text{Cpl}(\mu_0, \mu_1)} \int c(x, y)q(dx, dy).$$

Minimizers are called optimal couplings and, therefore, this family of problems is commonly called optimal transport problems. A natural and interesting question is when do these two minimization problems coincide, i.e. when is the or an optimal coupling given by a transportation map. In [4], Brenier showed using ideas from fluid dynamics that on Euclidean space with cost function  $c(x, y) = |x - y|^2$  there is always a unique optimal transportation map as soon as  $\mu_0$  is absolutely continuous with respect to the Lebesgue measure. Soon after, McCann [9] generalized this result to Riemannian manifolds with more general cost functions including convex functions of the distance. By now, this result is shown in a wide class of settings, for non-decreasing strictly convex functions of the distance in Alexandrov spaces [3], for squared distance on the Heisenberg group [2], and recently for the squared distance on  $\text{CD}(K, N)$  and  $\text{CD}(K, \infty)$  spaces by Gigli [5] and for squared distance cost by Rajala and Ambrosio in a metric Riemannian like framework [1].

In this paper we show existence and uniqueness of optimal transport maps on non-branching metric measure spaces satisfying a measure contraction property for cost functions of the form  $c(x, y) = h(d(x, y))$ , with  $h$  strictly convex and non-decreasing. In particular, our result recovers most of the previously mentioned results and in some cases also extends them. For example, Juillet [7] shows that the Heisenberg group satisfies the measure contraction property. Thus, our result extends the previously known results on the Heisenberg group to the case of non-decreasing strictly convex cost functions. As a drawback of the generality we do not get structural results on the transport map, as being a gradient of a nice function.

**1.1. Notation and main result.** Let  $(X, d, m)$  be a non-branching Polish metric measure space. Let  $\mathcal{G}(X)$  be the set of geodesics endowed with the uniform topology inherited from  $C([0, 1]; X)$ . Being a closed subset of  $C([0, 1]; X)$ , it is Polish. For  $t \in [0, 1]$  consider the map  $e_t : \mathcal{G}(X) \rightarrow X$  the evaluation at time  $t$  defined by  $e_t(\gamma) = \gamma_t$ . For a subset  $A \subset X$  and a point  $x \in X$  the  $t$ -intermediate points between  $A$  and  $x$  are defined as

$$A_t := e_t(\{\gamma \in \mathcal{G}(X) : \gamma_0 \in A, \gamma_1 = x\}).$$

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*Key words and phrases.* optimal transport; existence of maps; uniqueness of maps; measure contraction property.

We will assume that  $(X, d, m)$  satisfies  $\text{MCP}(K, N)$ , the so-called Measure Contraction property, for  $K \in \mathbb{R}$  and  $N \in \mathbb{N}$  with  $N \geq 2$ . The measure contraction property ensures the existence of a continuous function  $f : [0, 1] \rightarrow [0, 1]$  with  $f(0) = 1$  such that

$$m(A_t) \geq f(t) \cdot m(A).$$

We omit the precise definition of  $f$  and especially its dependence on  $K$  and  $N$  as we will only use the stated property. It is worth noting that  $\text{MCP}(K, N)$  is implied by  $\text{CD}(K, N)$ , see [13]. For more information on  $\text{MCP}(K, N)$  we refer to [11] and [13].

Let  $\mu_0, \mu_1$  be probability measures over  $X$ . We study the following minimization problem

$$\min_{T_{\#}\mu_0=\mu_1} \int h(d(x, T(x))) \mu_0(dx),$$

where  $h : [0, \infty) \rightarrow [0, \infty)$  is a strictly convex and non-decreasing map. In the sequel, we will often denote the cost function  $h \circ d$  just with  $c$ . Let  $\Pi(\mu_0, \mu_1)$  be the set of transference plans, i.e.

$$\Pi(\mu_0, \mu_1) := \{\pi \in \mathcal{P}(X \times X) : (P_1)_{\#}\pi = \mu_0, (P_2)_{\#}\pi = \mu_1\},$$

where  $P_i : X \times X \rightarrow X$  is the projection map on the  $i$ -th component,  $P_i(x_1, x_2) = x_i$  for  $i = 1, 2$ . We assume that  $\mu_0$  and  $\mu_1$  have finite  $c$ -transport distance in the sense that

$$\inf \left\{ \int_{X \times X} h(d(x, y)) \pi(dxdy) : \pi \in \Pi(\mu_0, \mu_1) \right\} < \infty.$$

Recall that a transference plan  $\pi \in \Pi(\mu_0, \mu_1)$  is said to be  $c$ -cyclically monotone if there exists  $\Gamma$  so that  $\pi(\Gamma) = 1$  and for every  $N \in \mathbb{N}$  and every  $(x_1, y_1), \dots, (x_N, y_N) \in \Gamma$  it holds

$$\sum_{i=1}^N c(x_i, y_i) \leq \sum_{i=1}^N c(x_{i+1}, y_i),$$

with  $x_{N+1} = x_1$ .

We will prove that, if  $\mu_0 \ll m$  any  $c$ -cyclically monotone plan  $\pi$  is induced by a map  $T : X \rightarrow X$ . With  $\pi$  induced by a map we mean that  $\pi = (id, T)_{\#}\mu_0$ , i.e. the two minimization problems (1.1) and (1.2) coincide. A direct Corollary of this result is the uniqueness of the optimal coupling.

More precisely, we will show that branching at starting points does not happen almost surely. The key idea is to approximate the  $c$ -cyclically monotone set on which the optimal coupling is concentrated.

## 2. EVOLUTION ESTIMATES

The probability measures  $\mu_0, \mu_1$  are fixed. Since we have to prove a local property, we can assume that  $\text{supp}(\mu_0), \text{supp}(\mu_1) \subset K$  with  $K$  compact. Then by standard results in optimal transportation, there exists a couple of Kantorovich potential  $(\varphi, \varphi^c)$  such that if

$$\Gamma := \{(x, y) \in X \times X : \varphi(x) + \varphi^c(y) = c(x, y)\},$$

then transport plan  $\pi$  is optimal iff  $\pi(\Gamma) = 1$  (e.g. see Theorem 5.10 in [14]).

We start by proving the standard property of geodesics belonging to the support of the optimal dynamical transference plan  $\eta$ : they cannot meet at the same time  $t$  if  $t \neq 0, 1$ . For existence results and details on dynamical transference plans we refer to [14] chapter 7.

**Lemma 2.1.** *Let  $(x_0, y_0), (x_1, y_1) \in \Gamma$  be two distinct points. Then for any  $t \in (0, 1)$ ,*

$$d(x_0(t), x_1(t)) > 0,$$

where  $x_i(t)$  is any  $t$ -intermediate point between  $x_i$  and  $y_i$ , for  $i = 0, 1$ .

*Proof.* Assume by contradiction the existence of  $x_0(t) = x_1(t) \in X$ ,  $t$ -intermediate points of  $(x_0, y_0)$  and  $(x_1, y_1)$ , i.e.

$$d(x_0, x_0(t)) = td(x_0, y_0), \quad d(x_0(t), y_0) = (1-t)d(x_0, y_0),$$

and

$$d(x_1, x_1(t)) = td(x_1, y_1), \quad d(x_1(t), y_1) = (1-t)d(x_1, y_1).$$

*Case 1:*  $d(x_0, y_0) \neq d(x_1, y_1)$ . Then

$$h(d(x_0, y_1)) + h(d(x_1, y_0)) \leq h(d(x_0, x_0(t)) + d(x_1(t), y_1)) + h(d(x_1, x_1(t)) + d(x_0(t), y_0))$$

$$\begin{aligned}
&< th(d(x_0, y_0)) + (1-t)h(d(x_1, y_1)) \\
&\quad + th(d(x_1, y_1)) + (1-t)h(d(x_0, y_0)) \\
&= h(d(x_0, y_0)) + h(d(x_1, y_1)).
\end{aligned}$$

Where between the first and the second line we have used the strict convexity of  $h$ . From  $c$ -cyclical monotonicity we have a contradiction.

*Case 2.*  $d(x_0, y_0) = d(x_1, y_1)$ . Let  $\gamma^0, \gamma^1 \in \mathcal{G}(X)$  be such that

$$\gamma_0^0 = x_0, \quad \gamma_t^0 = x_0(t), \quad \gamma_1^0 = y_0, \quad \gamma_0^1 = x_1, \quad \gamma_t^1 = x_1(t), \quad \gamma_1^1 = y_1,$$

and define the curve  $\gamma : [0, 1] \rightarrow X$  by

$$\gamma_t := \begin{cases} \gamma_s^0, & s \in [0, t] \\ \gamma_s^1, & s \in [t, 1]. \end{cases}$$

Then  $\gamma$  is a geodesic different from  $\gamma^0$  but coinciding with it on the non trivial interval  $[0, t]$ . Since this is a contradiction with the non-branching assumption, the claim is proved.  $\square$

For any closed set  $\Lambda \subset X \times X$  we can now consider the associated evolution map. For every  $t \in [0, 1]$  and every  $A \subset X$  compact set

$$A_{t, \Lambda} := e_t((e_0, e_1)^{-1}((A \times X) \cap \Lambda)).$$

Note that by the Arzelà-Ascoli Theorem,  $A_{t, \Lambda}$  is a compact set.

**Theorem 2.2.** *There exists a continuous map  $f : [0, 1] \rightarrow [0, 1]$  with  $f(0) = 1$  such that for any  $\Lambda \subset \Gamma$  compact the following inequality holds:*

$$(2.1) \quad m(A_{t, \hat{\Lambda}}) \geq f(t)m(A), \quad \forall t \in [0, 1],$$

for any  $A \subset P_1(\Lambda)$  compact set, where  $\hat{\Lambda} := (P_1(\Lambda) \times P_2(\Lambda)) \cap \Gamma$ .

*Proof. Step 1.* Let  $\{y_i\}_{i \in \mathbb{N}} \subset P_2(\Lambda)$  be a dense set in  $P_2(\Lambda)$ .

Consider the following family of sets: for  $n \in \mathbb{N}$  and  $i \leq n$

$$E_n(i) := \{x \in P_1(\Lambda) : c(x, y_i) - \varphi^c(y_i) \leq c(x, y_j) - \varphi^c(y_j), j = 1, \dots, n\}.$$

If we now consider

$$\Lambda_n := \bigcup_{i=1}^n E_n(i) \times \{y_i\},$$

it is straightforward to check that  $P_1(\Lambda_n) = P_1(\Lambda)$  and  $\Lambda_n$  is  $c$ -cyclically monotone. Indeed, for any  $(x_1, y_1), \dots, (x_m, y_m) \in \Lambda_n$ , by definition it holds

$$c(x_i, y_i) - \varphi^c(y_i) \leq c(x_i, y_{i+1}) - \varphi^c(y_{i+1}), \quad i = 1, \dots, m.$$

Taking the sum over  $i$ , the property follows.

By MCP( $K, N$ ) there exists a continuous map  $f : [0, 1] \rightarrow [0, 1]$  with  $f(0) = 1$ , independent of the sequence  $\{y_i\}_{i \in \mathbb{N}}$  and of  $n$ , such that for any  $A \subset P_1(\Lambda)$  compact it holds

$$m((A \cap E_n(i))_t) \geq f(t)m(A \cap E_n(i)),$$

where  $(A \cap E_n(i))_t = (A \cap E_n(i))_{t, E_n(i) \times \{y_i\}}$ . Note that since  $A = \bigcup_{i \leq n} A \cap E_n(i)$  it follows

$$\begin{aligned}
A_{t, \Lambda_n} &= e_t((e_0, e_1)^{-1}((A \times X) \cap \Lambda_n)) \\
&= \bigcup_{i \leq n} e_t((e_0, e_1)^{-1}(((A \cap E_n(i)) \times X) \cap \Lambda_n)) \\
&= \bigcup_{i \leq n} (A \cap E_n(i))_{t, \Lambda_n} \\
&\supset \bigcup_{i \leq n} (A \cap E_n(i))_{t, E_n(i) \times \{y_i\}}.
\end{aligned}$$

Moreover from Lemma 2.1 it holds

$$(A \cap E_n(i))_t \cap (A \cap E_n(j))_t = \emptyset, \quad i \neq j,$$

for all  $t \in (0, 1)$ .

Then it holds:

$$\begin{aligned}
 m(A_{t, \Lambda_n}) &\geq m\left(\bigcup_{i=1}^n (A \cap E_n(i))_{t, E_n(i) \times \{y_i\}}\right) = \sum_{i=1}^n m((A \cap E_n(i))_t) \\
 &\geq f(t) \sum_{i=1}^n m(A \cap E_n(i)) \\
 &\geq f(t) m\left(\bigcup_{i=1}^n A \cap E_n(i)\right) \\
 (2.2) \quad &= f(t) m(A).
 \end{aligned}$$

*Step 2.* Note that for every  $n \in \mathbb{N}$ ,  $\Lambda_n \subset \text{supp}(\mu_0) \times \text{supp}(\mu_1)$  and the latter, by assumption, is a subset of  $K \times K$ . Since the space of closed subsets of  $K \times K$  endowed with the Hausdorff metric  $(\mathcal{C}(K \times K), d_{\mathcal{H}})$  is a compact space, there exists a subsequence  $\{\Lambda_{n_k}\}_{k \in \mathbb{N}}$  and  $\Theta \subset K \times K$  compact such that

$$\lim_{k \rightarrow \infty} d_{\mathcal{H}}(\Lambda_{n_k}, \Theta) = 0.$$

Since the sequence  $\{y_i\}_{i \in \mathbb{N}}$  is dense in  $P_2(\Lambda)$  and  $\Lambda \subset \Gamma$  is compact, by definition of  $E_n(i)$ , necessarily for every  $(x, y) \in \Theta$  it holds

$$\varphi(x) + \varphi^c(y) = c(x, y), \quad x \in P_1(\Lambda), \quad y \in P_2(\Lambda).$$

Hence  $\Theta \subset (P_1(\Lambda) \times P_2(\Lambda)) \cap \Gamma = \hat{\Lambda}$ .

To conclude the proof we have to observe that also  $d_{\mathcal{H}}(A_{t, \Lambda_{n_k}}, A_{t, \Theta})$  converges to 0 as  $k \rightarrow \infty$ . Then

$$m(A_{t, \Theta}) \geq \limsup_{k \rightarrow \infty} m(A_{t, \Lambda_{n_k}}).$$

Indeed, since  $A_{t, \Theta}$  is a compact set, it follows that if  $A_{t, \Theta}^\varepsilon = \{x \in X : d(x, A_{t, \Theta}) \leq \varepsilon\}$ , then for  $k$  sufficiently big  $A_{t, \Lambda_{n_k}} \subset A_{t, \Theta}^\varepsilon$  and  $m(A_{t, \Lambda_{n_k}})$  converges to  $m(A_{t, \Theta})$ .

Then

$$m(A_{t, \hat{\Lambda}}) \geq \limsup_{k \rightarrow \infty} m(A_{t, \Lambda_{n_k}}) \geq f(t) m(A),$$

and the claim follows.  $\square$

### 3. EXISTENCE OF OPTIMAL MAPS

In this section we show that branching at starting points does not happen almost surely.

**Lemma 3.1.** *Let  $\Lambda_1, \Lambda_2 \subset \Gamma$  be compact sets such that*

- i)  $P_1(\Lambda_1) = P_1(\Lambda_2)$ ;
- ii)  $P_2(\Lambda_1) \cap P_2(\Lambda_2) = \emptyset$ .

*Then  $m(P_1(\Lambda_1)) = m(P_1(\Lambda_2)) = 0$ .*

*Proof.* Note that since  $P_2(\Lambda_1) \cap P_2(\Lambda_2) = \emptyset$ , necessarily  $\hat{\Lambda}_1 \cap \hat{\Lambda}_2 = \emptyset$ . Hence from Lemma 2.1, for every  $A \subset P_1(\Lambda_1) = P_1(\Lambda_2)$

$$A_{t, \hat{\Lambda}_1} \cap A_{t, \hat{\Lambda}_2} = \emptyset,$$

for every  $t \in (0, 1)$ . Then let  $A := P_1(\Lambda_1) = P_1(\Lambda_2)$  and recall that as  $t \rightarrow 0$  the sets  $A_{t, \Lambda_1}$  and  $A_{t, \Lambda_2}$  both converge in Hausdorff topology to  $A$ . Put  $A^\varepsilon = \{x : d(x, A) \leq \varepsilon\}$ . Then it follows from Theorem 2.2 that

$$\begin{aligned}
 m(A) &= \limsup_{\varepsilon \rightarrow 0} m(A^\varepsilon) \geq \limsup_{t \rightarrow 0} m(A_{t, \Lambda_1} \cup A_{t, \Lambda_2}) \\
 &= \limsup_{t \rightarrow 0} m(A_{t, \Lambda_1}) + m(A_{t, \Lambda_2}) \\
 &\geq 2m(A) \limsup_{t \rightarrow 0} f(t) = 2m(A).
 \end{aligned}$$

Hence, necessarily  $m(P_1(\Lambda_1)) = m(P_1(\Lambda_2)) = m(A) = 0$ , and the claim follows.  $\square$

We will use the following notation  $\Gamma(x) := (\{x\} \times X) \cap \Gamma$  and given a set  $\Theta \subset X \times X$  we say that  $T$  is a selection of  $\Theta$  if  $T : P_1(\Theta) \rightarrow X$  is  $m$ -measurable and  $\text{graph}(T) \subset \Theta$ .

**Proposition 3.2.** *Assume  $\mu_0 \ll m$ . Consider the sets*

$$E := \{x \in P_1(\Gamma) : \Gamma(x) \text{ is not a singleton}\}, \quad \Gamma_E := \Gamma \cap (E \times X).$$

*Then for any selection  $T$  of  $\Gamma_E$  and every  $\pi \in \Pi(\mu_0, \mu_1)$  with  $\pi(\Gamma) = 1$  it holds*

$$\pi(\Gamma_E \setminus \text{graph}(T)) = 0.$$

*Proof. Step 1.* Suppose by contradiction the existence of  $\pi \in \Pi(\mu_0, \mu_1)$  with  $\pi(\Gamma) = 1$  and of a selection  $T$  of  $\Gamma_E$  such that

$$\pi(\Gamma_E \setminus \text{graph}(T)) = \alpha > 0.$$

By inner regularity of compact sets, to prove the complete statement it is enough to prove it under the additional assumptions that  $E$  is compact and  $T$  is continuous.

Note that

$$\Gamma_E \setminus \text{graph}(T) = \bigcup_{n=1}^{\infty} \{(x, y) \in \Gamma_E : d(y, T(x)) \geq 1/n\}.$$

Hence, there exists  $n \in \mathbb{N}$  such that

$$\pi(\{(x, y) \in \Gamma_E : d(y, T(x)) \geq 1/n\}) \geq \alpha' > 0.$$

*Step 2.* From continuity of  $T$ , the set  $\{(x, y) \in \Gamma_E : d(y, T(x)) \geq 1/n\}$  is compact, hence there exists  $N \in \mathbb{N}$  and points  $(x_1, y_1), \dots, (x_N, y_N) \in \{(x, y) \in \Gamma_E : d(y, T(x)) \geq 1/n\}$  such that

$$\{(x, y) \in \Gamma_E : d(y, T(x)) \geq 1/n\} \subset \bigcup_{i=1}^N B_{\frac{1}{5n}}((x_i, y_i)),$$

where on the product space  $X \times X$  we consider the usual distance  $(d_1^2 + d_2^2)^{1/2}$ . Hence there exists a couple  $(x_i, y_i)$  such that

$$\pi(B_{1/5n}((x_i, y_i)) \cap \{(x, y) \in \Gamma_E : d(y, T(x)) \geq 1/n\}) \geq \frac{\alpha'}{N}.$$

Since the measure  $m$  restricted to  $K$ , as defined at the beginning of section 2 is doubling (see remark 5.3 in [13]), we can also assume  $x_1, \dots, x_N$  to be points of Lebesgue  $m$ -density one in  $P_1(\{(x, y) \in \Gamma_E : d(y, T(x)) \geq 1/n\})$  (e.g. see Chapter 1 of [6]). Note indeed that the latter has positive  $m$ -measure.

From the continuity of  $T$  it follows the existence of  $\delta > 0$  so that if  $d(x, x_i) \leq \delta$  then  $d(T(x), T(x_i)) \leq 1/5n$ . Hence

$$T(B_\delta(x_i)) \cap P_2(B_{1/5n}((x_i, y_i))) = \emptyset,$$

indeed for any  $y \in P_2(B_{1/5n}((x_i, y_i)))$  and  $x \in B_\delta(x_i)$

$$\begin{aligned} d(y, T(x)) &\geq d(y_i, T(x)) - d(y, y_i) \\ &\geq d(y_i, T(x_i)) - d(T(x_i), T(x)) - d(y, y_i) \\ &\geq \frac{1}{n} - \frac{1}{5n} - \frac{1}{5n} = \frac{3}{5n}. \end{aligned}$$

*Step 3.* So consider

$$\Xi_1 := \text{graph}(T) \cap (\bar{B}_\delta(x_i) \times X), \quad \Xi_2 := \bar{B}_{\frac{1}{5n}}((x_i, y_i)) \cap \{(x, y) \in \Gamma_E : d(y, T(x)) \geq 1/n\},$$

where with  $\bar{B}$  we intend the closed ball. By construction  $\Xi_1, \Xi_2 \subset \Gamma$  and from *Step 2*.

$$P_2(\Xi_1) \cap P_2(\Xi_2) = \emptyset.$$

Then define  $\lambda := P_1(\Xi_1) \cap P_1(\Xi_2)$  and since  $x_i$  is a point of Lebesgue  $m$ -density one for  $P_1(\{(x, y) \in \Gamma_E : d(y, T(x)) \geq 1/n\})$ , it follows that  $m(\lambda) > 0$ . Hence the sets

$$\Lambda_1 := \Xi_1 \cap (\lambda \times X), \quad \Lambda_2 := \Xi_2 \cap (\lambda \times X)$$

are so that:  $P_1(\Lambda_1) = P_1(\Lambda_2)$ ,  $P_2(\Lambda_1) \cap P_2(\Lambda_2) = \emptyset$  while  $m(P_1(\Lambda_1)) > 0$ . Since this is in contradiction with Lemma 3.1, the claim is proved.  $\square$

The proof of the following is now a straightforward corollary of what we proved so far.

**Theorem 3.3.** *For any  $\pi \in \Pi(\mu_0, \mu_1)$  such that  $\pi(\Gamma) = 1$  there exists an  $m$ -measurable map  $T : X \rightarrow X$  such that*

$$\pi(\text{graph}(T)) = 1.$$

*Proof.* Let  $\pi \in \Pi(\mu_0, \mu_1)$  be any transference plan so that  $\pi(\Gamma) = 1$ . As for Proposition 3.2, consider the sets

$$E := \{x \in P_1(\Gamma) : \Gamma(x) \text{ is not a singleton}\}, \quad \Gamma_E := \Gamma \cap (E \times X).$$

Since

$$\Gamma_E = P_{12}(\{(x, y, z, w) \in \Gamma \times \Gamma : d(x, z) = 0, d(y, w) > 0\}),$$

the set  $\Gamma_E$  is an analytic set. For the definition of analytic set, see Chapter 4 of [12]. We can then use Von Neumann Selection Theorem for analytic sets, see Theorem 5.5.2 of [12], to obtain a map  $T : E \rightarrow X$ ,  $\mathcal{A}$ -measurable, where  $\mathcal{A}$  is the  $\sigma$ -algebra generated by analytic sets, so that  $(x, T(x)) \in \Gamma_E$ .

Then Proposition 3.2 implies that

$$\pi \llcorner_{\Gamma_E} = (Id, T)_\# \mu_0 \llcorner_E.$$

Since on  $\Gamma \setminus \Gamma_E$   $\pi$  is already supported on a graph, the claim follows.  $\square$

This directly implies

**Corollary 3.4.** *There is a unique optimal transport map.*

*Proof.* The last theorem shows that every optimal coupling is induced by a transport map. As the set of all optimal couplings is convex this directly implies the uniqueness.  $\square$

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